

Calculation Guidance for Primary Schools

One of the first projects undertaken by the Maths Hubs programme, launched in summer 2014, was the England-China school exchange, involving primary teachers in the school year 2014-15, and secondary teachers in the current school year, 2015-16. In both these exchanges, English teachers visit Shanghai for a fortnight and are immersed in a number of schools there, and then their exchange partner teachers from those schools come to England for a month and teach maths in the English teachers' schools alongside their partners.

On 23 and 24 June 2015, the 71 teachers from the 47 lead primary schools (the LPS) who participated in the first year of the exchange gathered at the National College for Teaching and Leadership; together, they had hosted 55 Shanghai exchange teachers in their schools. They brought to Nottingham and shared with each other what they had learned from the exchange so far, and they also set out their plans for embedding and extending their knowledge in the coming academic year, in their own and also in their local partner schools.

One session of the conference focused in detail on “calculation guidance”; the purpose was to produce a list of recommendations and effective practice teaching ideas developed during and after the exchange visits. This document summarises and synthesises the discussions that took place and the reflections that were shared by the LPS teachers, and also draws on the two written reports submitted by each LPS during 2014-15. This is not intended to be a calculation policy as such; rather, it could sit alongside a school's existing policy, and the ideas captured here (which are indicative and not exhaustive) could inform and enhance teaching across all primary key stages.

The following priority areas have been identified. Each of these is discussed in more detail with examples below:

- [Develop children's fluency with basic number facts](#)
- [Develop children's fluency in mental calculation](#)

- [Develop children's fluency in the use of written methods](#)
- [Develop children's understanding of the = symbol](#)
- [Teach inequality alongside teaching equality](#)
- [Don't count, calculate](#)
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- [Expose mathematical structure and work systematically](#)
- [Move between the concrete and the abstract](#)
- [Contextualise the mathematics](#)
- [Use questioning to develop mathematical reasoning](#)
- [Expect children to use correct mathematical terminology and speak in full sentences](#)
- [Identify difficult points](#)

Develop children's fluency with basic number facts

Fluent computational skills are dependent on accurate and rapid recall of basic number bonds to 20 and times-tables facts. The LPS schools have found that spending a short time everyday on these basic facts quickly leads to improved fluency. This can be done using simple whole class chorus chanting. The LPS teachers are clear that this is not meaningless rote learning; rather, this is an important step to developing conceptual understanding through identifying patterns and relationships between the tables (for example, that the products in the $6\times$ table are double the products in the $3\times$ table). They have found that this has helped children develop a strong sense of number relationships, an important prerequisite for procedural fluency.

Children in Shanghai learn their multiplication tables in this order to provide opportunities to make connections:

x10	x5	x2	x4	x8	x3	x6	x9	x7
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Develop children's fluency in mental calculation

Efficiency in calculation requires having a variety of mental strategies. In particular the Shanghai teachers who participated in the exchange emphasised the importance of 10 and partitioning numbers to bridge through 10. For example:

$$9 + 6 = 9 + 1 + 5 = 10 + 5 = 15.$$

The Shanghai teachers referred to "magic 10". It is helpful to make a 10 as this makes the calculation easier.

Develop fluency in the use of formal written methods

Teaching column methods for calculation provides the opportunity to develop both procedural and conceptual fluency. The LPS teachers noted that the Shanghai teachers ensured that children understood the structure of the mathematics presented in the algorithms, with a particular emphasis on place value. They saw base ten apparatus being used and illustrated in textbooks to support the development of fluency and understanding.

Informal methods of recording calculations are an important stage to help children develop fluency with formal methods of recording. A noticeable difference, however, that the LPS teachers observed in Shanghai is that these were only used for a short period, to help children understand the internal logic of formal methods of recording calculations. They are stepping stones to formal written methods. Here is an example from a Shanghai textbook:

$$23 \times 4 = ?$$

x	2	3		
	1	2		---
				4×3
	8	0		---
				4×20
	9	2		

x	2	3		
	1	4		
	9	2		

Stepping stones to formal written methods

Develop children's understanding of the = symbol

The symbol = is an assertion of equivalence. If we write:

$$3 + 4 = 6 + 1$$

then we are saying that what is on the left of the = symbol is necessarily equivalent to what is on the right of the symbol. But many children interpret = as being simply an instruction to evaluate a calculation, as a result of always seeing it used thus:

$$3 + 4 =$$

$$5 \times 7 =$$

$$16 - 9 =$$

If children only think of = as meaning "work out the answer to this calculation" then they are likely to get confused by empty box questions such as:

$$3 + \square = 8$$

Later they are very likely to struggle with even simple algebraic equations, such as:

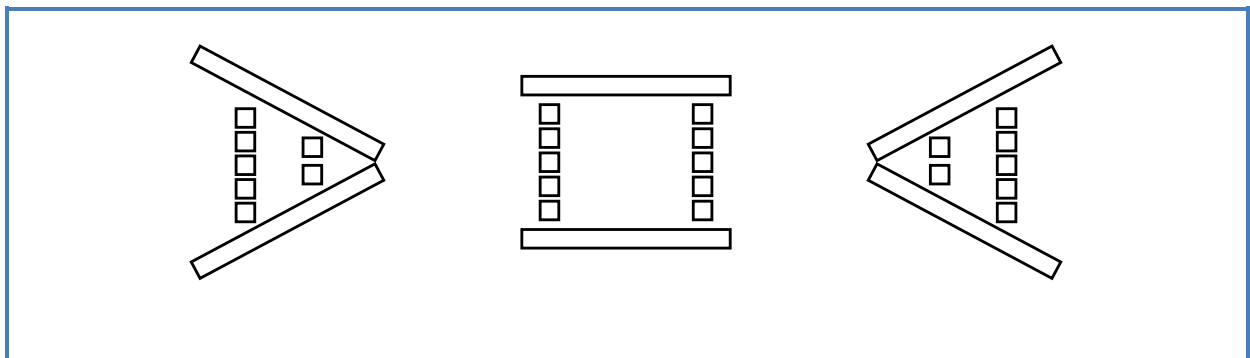
$$3y = 18$$

One way to model equivalence such as $2 + 3 = 5$ is to use balance scales.

Chinese textbooks vary the position of the = symbol and include empty box problems from Grade 1 (equivalent to Year 2 in England) to deepen children's understanding of the = symbol.

Teach inequality alongside teaching equality

To help young children develop their understanding of equality, they also need to develop understanding of inequality. Some of the LPS teachers have experimented with teaching inequality before, or at the same time as, equality (as they observed in lessons in Shanghai). One way to introduce the < and > signs is to use rods and cubes to make a concrete and visual representations such as:



to show that 5 is greater than 2 ($5 > 2$), 5 is equal to 5 ($5 = 5$), and 2 is less than 5 ($2 < 5$).

Balance scales can also be used to represent inequality.

Incorporating both equality and inequality into examples and exercises can help children develop their conceptual understanding. For example, in this empty box problem children have to decide whether the missing symbol is $<$, $=$ or $>$:

$$5 + 7 \square 5 + 6$$

An activity like this also encourages children to develop their mathematical reasoning: “I know that 7 is greater than 6, so 5 plus 7 must be greater than 5 plus 6”.

Asking children to decide if number sentences are true or false also helps develop mathematical reasoning. For example, in discussing this statement:

$$4 + 6 + 8 > 3 + 7 + 9$$

a child might reason that “4 plus 6 and 3 plus 7 are both 10. But 8 is less than 9. Therefore $4 + 6 + 8$ must be less than $3 + 7 + 9$, not more than $3 + 7 + 9$ ”.

In both these examples the numbers have been deliberately chosen to allow the children to establish the answer without actually needing to do the computation. This emphasises further the importance of mathematical reasoning.

Don't count, calculate

Young children benefit from being helped at an early stage to start calculating, rather than relying on ‘counting on’ as a way of calculating. For example, with a sum such as:

$$4 + 7 =$$

Rather than starting at 4 and counting on 7, children could use their knowledge and bridge to 10 to deduce that because $4 + 6 = 10$, so $4 + 7$ must equal 11.

Look for pattern and make connections

The Shanghai teachers used a great many visual representations of the mathematics and some concrete resources. Understanding, however, does not happen automatically, children need to reason by and with themselves and make their own connections. The Shanghai teachers talked about getting children into good habits from Year 1 in terms of reasoning and looking for pattern and connections in the mathematics. The question “What’s the same, what’s different?” is used frequently to make comparisons. For example “*What’s the same, what’s different between the three times table and the six times table?*”

Use intelligent practice

Chinese children do engage in a significant amount of practice of mathematics through class- and homework exercises. However, *in designing [these] exercises, the teacher is advised to avoid mechanical repetition and to create an appropriate path for practising the thinking process with increasing creativity* (Gu, 1991). The practice that Chinese children engage in provides the opportunity to develop both procedural and conceptual fluency. Children are required to reason and make connections between calculations. The connections made improve their fluency.

For example:

$2 \times 3 =$

$6 \times 7 =$

$9 \times 8 =$

$2 \times 30 =$

$6 \times 70 =$

$9 \times 80 =$

$2 \times 300 =$

$6 \times 700 =$

$9 \times 800 =$

$20 \times 3 =$

$60 \times 7 =$

$90 \times 8 =$

$200 \times 3 =$

$600 \times 7 =$

$900 \times 8 =$

Shanghai Textbook Grade 2 (aged 7/8)

Use empty box problems

Empty box problems are a powerful way to help children develop a strong sense of number through intelligent practice. They provide the opportunity for reasoning and finding easy ways to calculate. They enable children to practise procedures, whilst at the same time thinking about conceptual connections.

A sequence of examples such as

$$3 + \square = 8$$

$$3 + \square = 9$$

$$3 + \square = 10$$

$$3 + \square = 11$$

helps children develop their understanding that the = symbol is an assertion of equivalence, and invites children to spot the pattern and use this to work out the answers.

This sequence of examples does the same at a deeper level:

$$3 \times \square + 2 = 20$$

$$3 \times \square + 2 = 23$$

$$3 \times \square + 2 = 26$$

$$3 \times \square + 2 = 29$$

$$3 \times \square + 2 = 35$$

Children should also be given examples where the empty box represents the operation, for example:

$$4 \times 5 = 10 \square 10$$

$$6 \square 5 = 15 + 15$$

$$6 \square 5 = 20 \square 10$$

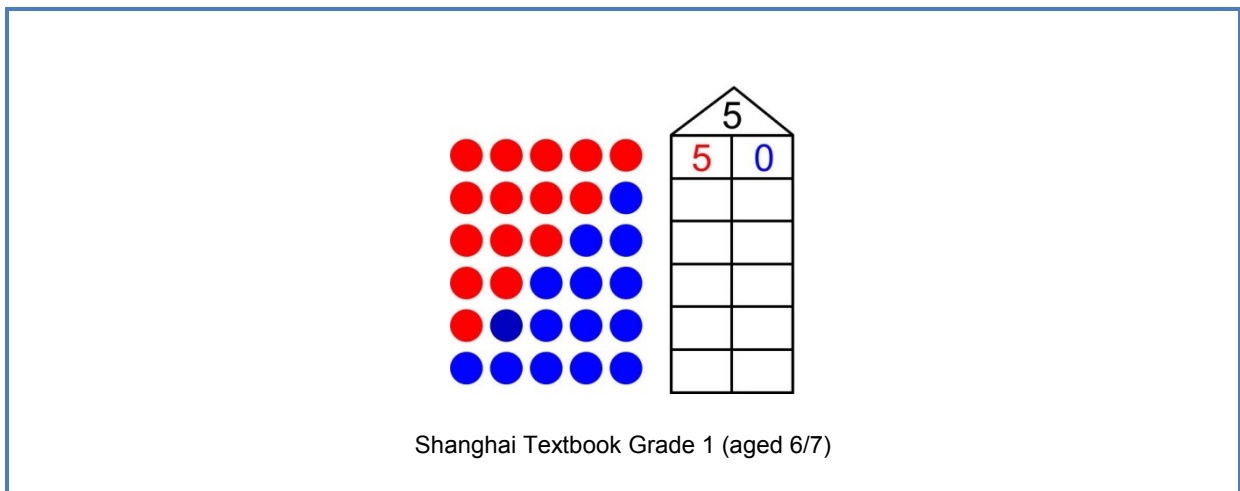
$$8 \square 5 = 20 \square 20$$

$$8 \square 5 = 60 \square 20$$

These examples also illustrate the careful use of variation to help children develop both procedural and conceptual fluency.

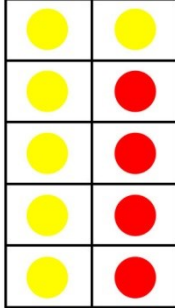
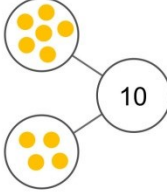
Expose mathematical structure and work systematically

Developing instant recall alongside conceptual understanding of number bonds to 10 is important. This can be supported through the use of images such as the example illustrated below:



The image lends itself to seeing pattern and working systematically and children can connect one number fact to another and be certain when they have found all the bonds to 5.

Using other structured models such as tens frames, part whole models or bar models can help children to reason about mathematical relationships.

 $6 + 4 = 10$ $4 + 6 = 10$ $10 - 4 = 6$ $10 - 6 = 4$ <p style="text-align: center;">Tens Frame</p>	 $6 + 4 = 10$ $4 + 6 = 10$ $10 - 4 = 6$ $10 - 6 = 4$ <p style="text-align: center;">Part Whole Model</p>	<table border="1" style="margin-left: auto; margin-right: auto;"> <tbody> <tr> <td colspan="2" style="text-align: center;">10</td> </tr> <tr> <td style="text-align: center;">6</td> <td style="text-align: center;">4</td> </tr> </tbody> </table> $6 + 4 = 10$ $4 + 6 = 10$ $10 - 4 = 6$ $10 - 6 = 4$ <p style="text-align: center;">Bar Model</p>	10		6	4
10						
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Connections between these models should be made, so that children understand the same mathematics is represented in different ways. Asking the question “What’s the same what’s different?” has the potential for children to draw out the connections.

Illustrating that the same structure can be applied to any numbers helps children to generalise mathematical ideas and build from the simple to more complex numbers, recognising that the structure stays the same; it is only the numbers that change.

For example:

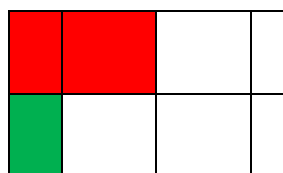
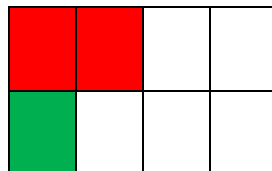
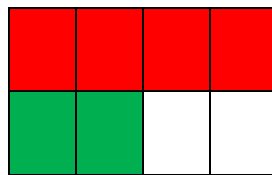
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Move between the concrete and the abstract

Children's conceptual understanding and fluency is strengthened if they experience concrete, visual and abstract representations of a concept during a lesson. Moving between the concrete and the abstract helps children to connect abstract symbols with familiar contexts, thus providing the opportunity to make sense of, and develop fluency in the use of, abstract symbols.

For example, in a lesson about addition of fractions children could be asked to draw

a picture to represent the sum $\frac{1}{4} + \frac{1}{8} = \frac{3}{8}$. Alternatively, or in a subsequent lesson, they could be asked to discuss which of three visual images correctly represents the sum, and to explain their reasoning:



Contextualise the mathematics

A lesson about addition and subtraction could start with this contextual story:

“There are 11 people on a bus. At the next stop 4 people get on. At the next stop 6 people get off. How many are now on the bus?”

This helps children develop their understanding of the concepts of addition and subtraction. But during the lesson the teacher should keep returning to the story. For example, if the children are thinking about this calculation

$$14 - 8$$





then the teacher should ask the children:

“What does the 14 mean? What does the 8 mean?”, expecting that children will answer:

“There were 14 people on the bus, and 8 is the number who got off.”

Then asking the children to interpret the meaning of the terms in a sum such as $7 + 7 = 14$ will give a good assessment of the depth of their conceptual understanding and their ability to link the concrete and abstract representations of mathematics

The four slides below are taken from a lesson delivered by one of the Shanghai teachers (Li Dong)

<p>Before then now</p>  <p>Write the number sentence that matches this story</p> <p>Slide 1</p>	<p>Before then now</p>  <p>Draw the middle picture and Write the number sentence that matches this story</p> <p>Slide 2</p>
<p>Before then now</p>  <p>$4 - 0 = 4$</p> <p>Slide 3</p>	<p>Before then now</p>  <p>Finish the story and write the number sentence</p> <p>Slide 4</p>

Notice how each activity varies. The children are asked to:

Slide 1: Start with the story (concrete) and write the number sentence (abstract).

Slide 2: Start with the story (concrete) and complete it. Then write the number sentence (abstract).

Slide 3: Start with the number sentence (abstract) and complete the story (concrete).

Slide 4: Start with part of the story, complete two elements of it (concrete with challenge) and then write the number sentence (abstract).

The children move between the concrete and the abstract and back to the concrete, with an increasing level of difficulty.

Use questioning to develop mathematical reasoning

Teachers' questions in mathematics lessons are often asked in order to find out whether children can give the right answer to a calculation or a problem. But in order to develop children's conceptual understanding and fluency there needs to be a strong and consistent focus on questioning that encourages and develops their mathematical reasoning.

This can be done simply by asking children to explain how they worked out a calculation or solved a problem, and to compare and contrast different methods that are described. The LPS teachers have found that children quickly come to expect that they need to explain and justify their mathematical reasoning, and they soon start to do so automatically – and enthusiastically. Some calculation strategies are more efficient and the LPS teachers noted that the Shanghai teachers scaffolded children's thinking to guide them to the most efficient methods, whilst at the same time valuing their own ideas.

Rich questioning strategies include:

- *“What's the same, what's different?”*

In this sequence of expressions, what stays the same each time and what's different?

$23 + 10$	$23 + 20$	$23 + 30$	$23 + 40$
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Discussion of the variation in these examples can help children to identify the relationship between the calculations and hence to use the pattern to calculate the answers.

- *“Odd one out”*

Which is the odd one out in this list of numbers: 24, 15, 16 and 22?

This encourages children to apply their existing conceptual understanding. Possible answers could be:

“15 is the odd one out because it’s the only odd number in the list.”

“16 is the odd one out because it’s the only square number in the list.”

“22 is the odd one out because it’s the only number in the list with exactly four factors.”

If children are asked to identify an ‘odd one out’ in this list of products:

$$24 \times 3 \quad 36 \times 4 \quad 13 \times 5 \quad 32 \times 2$$

they might suggest:

“ 36×4 is the only product whose answer is greater than 100.”

“ 13×5 is the only product whose answer is an odd number.”

- *“Here’s the answer. What could the question have been?”*

Children are asked to suggest possible questions that have a given answer. For example, in a lesson about addition of fractions, children could be asked to suggest possible ways to complete this sum:

$$\square + \square = \frac{3}{4}$$

- *Identify the correct question*

Here children are required to select the correct question:

A 3.5m plank of wood weighs 4.2 kg

The calculation was:

$$3.5 \div 4.2$$

Was the question:

- How heavy is 1m of wood?
- How long is 1kg of wood?

- True or False*

Children are given a series of equations are asked whether they are true or false:

$$4 \times 6 = 23 \quad 4 \times 6 = 6 \times 4 \quad 12 \div 2 = 24 \div 4 \quad 12 \times 2 = 24 \times 4$$

Children are expected to reason about the relationships within the calculations rather than calculate

- Greater than, less than or equal to >, <, or =*

$$3.4 \times 1.2 \bigcirc 3.4 \quad 5.76 \bigcirc 5.76 \div 0.4 \quad 4.69 \times 0.1 \bigcirc 4.69 \div 10$$

These types of questions are further examples of intelligent practice where conceptual understanding is developed alongside the development of procedural fluency. They also give pupils who are, to use Ofsted's phrase, *rapid graspers* the opportunity to apply their understanding in more complex ways.

Expect children to use correct mathematical terminology and to express their reasoning in complete sentences

The quality of children's mathematical reasoning and conceptual understanding is significantly enhanced if they are consistently expected to use correct mathematical terminology (e.g. saying 'digit' rather than 'number') and to explain their mathematical thinking in complete sentences.

I say, you say, you say, you say, we all say

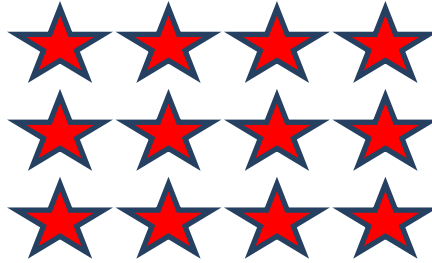
This technique enables the teacher to provide a sentence stem for children to communicate their ideas with mathematical precision and clarity. These sentence structures often express key conceptual ideas or generalities and provide a framework to embed conceptual knowledge and build understanding. For example:

If the rectangle is the whole, the shaded part is one third of the whole.

Having modelled the sentence, the teacher then asks individual children to repeat this, before asking the whole class to chorus chant the sentence. This provides children with a valuable sentence for talking about fractions. Repeated use helps to embed key conceptual knowledge.

Another example is where children fill in the missing parts of a sentence; varying the parts but keeping the sentence stem the same. For example:

There are 12 stars. $\frac{1}{3}$ of the stars is equal to 4 stars

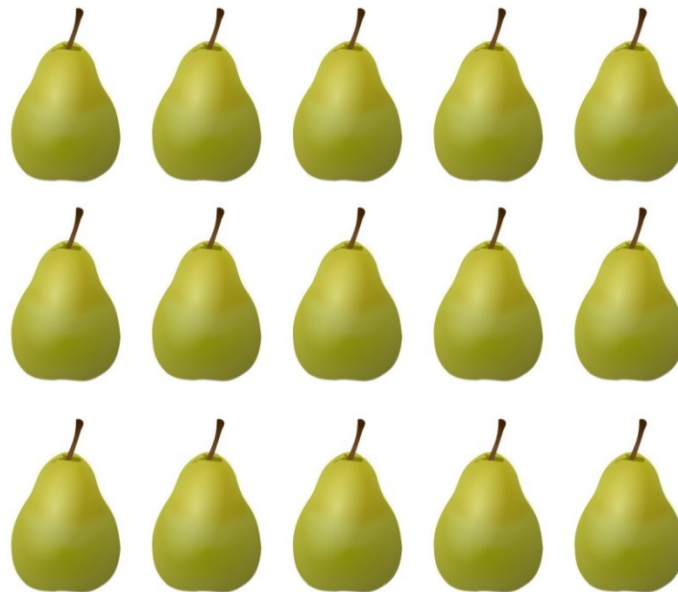


Children use the same sentence stem to express other relationships. For example:

There are 12 stars. $\frac{1}{4}$ of the stars is equal to 3 stars

There are 12 stars. $\frac{1}{2}$ of the stars is equal to 6 stars

Similarly:



There are 15 pears. $\frac{1}{3}$ of the pears is equal to 5 pears

There are 15 pears. $\frac{1}{5}$ of the pears is equal to 3 pears

When talking about fractions it is important to make reference to the whole and the part of the whole in the same sentence. The above examples help children to get into the habit of doing so.

Another example is where a mathematical generalisation or “*rule*” emerges within a lesson. For example:

When adding 10 to a number, the ones digit stays the same

This is repeated in chorus using the same sentence, which helps to embed the concept.

Identify difficult points

Difficult points need to be identified and anticipated when lessons are being designed and these need to be an explicit part of the teaching, rather than the teacher just responding to children’s difficulties if they happen to arise in the lesson. The teacher should be actively seeking to uncover possible difficulties because if one child has a difficulty it is likely that others will have a similar difficulty. Difficult points also give an opportunity to reinforce that we learn most by working on and through ideas with which we are not fully secure or confident. Discussion about difficult points can be stimulated by asking children to share thoughts about their own examples when these show errors arising from insufficient understanding. For example:

$$\frac{2}{14} - \frac{1}{7} = \frac{1}{7}$$

A visualiser is a valuable resource since it allows the teacher quickly to share a child’s thinking with the whole class.